All possible values for the angle theta

Remember that $\cos\theta$ is given by the x-value, and $\sin\theta$ is given by the y-value, of coordinate points along the unit circle.

$$x = \cos\theta$$

$$y = \sin\theta$$

Therefore, when we want to find all the values of θ that satisfy a trig equation, we need to look for angles along the unit circle. If we find an angle that makes the equation true, we need to remember to include all other coterminal angles, since those angles all have the same sine and cosine values.

Example

Solve the equation $\cos\theta = 0$.

Well, we know that $\cos 90^\circ = 0$ and $\cos 270^\circ = 0$, and that there is no other angle in the interval $[0^\circ, 360^\circ)$ whose cosine is equal to 0. Therefore, the solutions of the equation $\cos\theta = 0$ are

$$90^\circ,\ 90^\circ \pm 360^\circ,\ 90^\circ \pm 720^\circ,\ \ldots$$

and

$$270^\circ,\ 270^\circ \pm 360^\circ,\ 270 \pm 720^\circ,\ \ldots$$

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We can express the solution set of the equation $\cos\theta = 0$ as

$$\{90^\circ \pm (360^\circ)k : \ k = 0,1,2,\ldots\} \cup \{270^\circ \pm (360^\circ)k : \ k = 0,1,2,\ldots\}$$

That is, we can use an integer-valued variable, such as k, to list all the solutions very succinctly.

In radians, an angle of 90° is $\pi/2$ and an angle of 270° is $3\pi/2$. Therefore, in radians the solution set of the equation $\cos\theta = 0$ is

$$\left\{\frac{\pi}{2} \pm 2\pi k : \ k = 0,1,2,\ldots\right\} \cup \left\{\frac{3\pi}{2} \pm 2\pi k : \ \ k = 0,1,2,\ldots\right\}$$

Let's look at a different example.

Example

Solve the equation $\cos\theta = \sqrt{2}/2$.

We know that

$$\cos 45^\circ = \frac{\sqrt{2}}{2}$$

Thus one value of θ that solves the equation $\cos\theta = \sqrt{2}/2$ is 45°, but there are infinitely many angles θ whose terminal side coincides with the terminal side of 45°. These angles make up the set

$$\{45^\circ \pm (360^\circ)k : \ k = 0,1,2,\ldots\}$$

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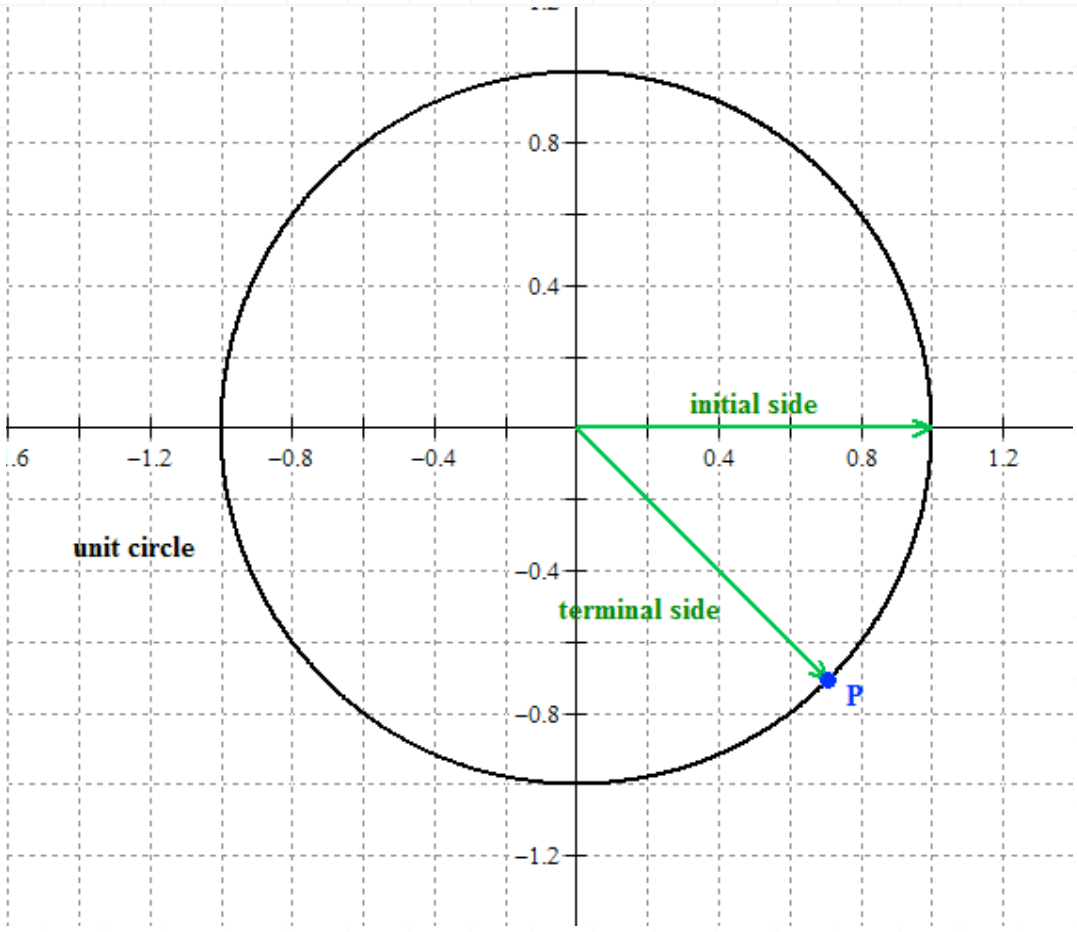
We know that $\cos\theta = \sqrt{2}/2$ is also true when $\theta = 315°$. That angle looks like



All the angles that are coterminal with $\theta = 315°$ make up the set

$$\{315° \pm (360°)k : \ k = 0,1,2,\ldots\}$$

Combining our results, we find that the solution set of the equation $\cos\theta = \sqrt{2}/2$ is

$$\{45° \pm (360°)k : \ k = 0,1,2,\ldots\} \cup \{315° \pm (360°)k : \ k = 0,1,2,\ldots\}$$

In radians, an angle of $45°$ is $\pi/4$ and an angle of $315°$ is $7\pi/4$. Therefore, if we wish to express the solution set of the equation $\cos\theta = \sqrt{2}/2$ in radians, we get

$$\left\{\frac{\pi}{4} \pm 2\pi k : \ k = 0,1,2,\ldots\right\} \cup \left\{\frac{7\pi}{4} \pm 2\pi k : \ k = 0,1,2,\ldots\right\}$$

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Next, let's consider solutions of an equation of the form $\cos\theta = a$ where a is a negative number. Then a is in the interval $[-1,0)$.

Example

Solve the equation $\cos\theta = -0.643$.

If an angle θ satisfies this equation, then $\cos\theta$ is negative, which means the x-coordinate is negative. Therefore, θ has to be in the second or third quadrant.

If we use a calculator to find the inverse cosine, we get

$$\cos\theta = -0.643$$

$$\arccos(\cos\theta) = \arccos(-0.643)$$

$$\theta = \arccos(-0.643)$$

$$\theta \approx 130^\circ$$

So one set of angles that satisfy the equation is

$$\{130^\circ \pm (360^\circ)k : \ k = 0,1,2,\ldots\}$$

We can find the other set of angles by symmetry. If 130° gives us one set of angles, and 130° is 50° "above" the x-axis, then the other set of angles is at 50° "below" the x-axis.

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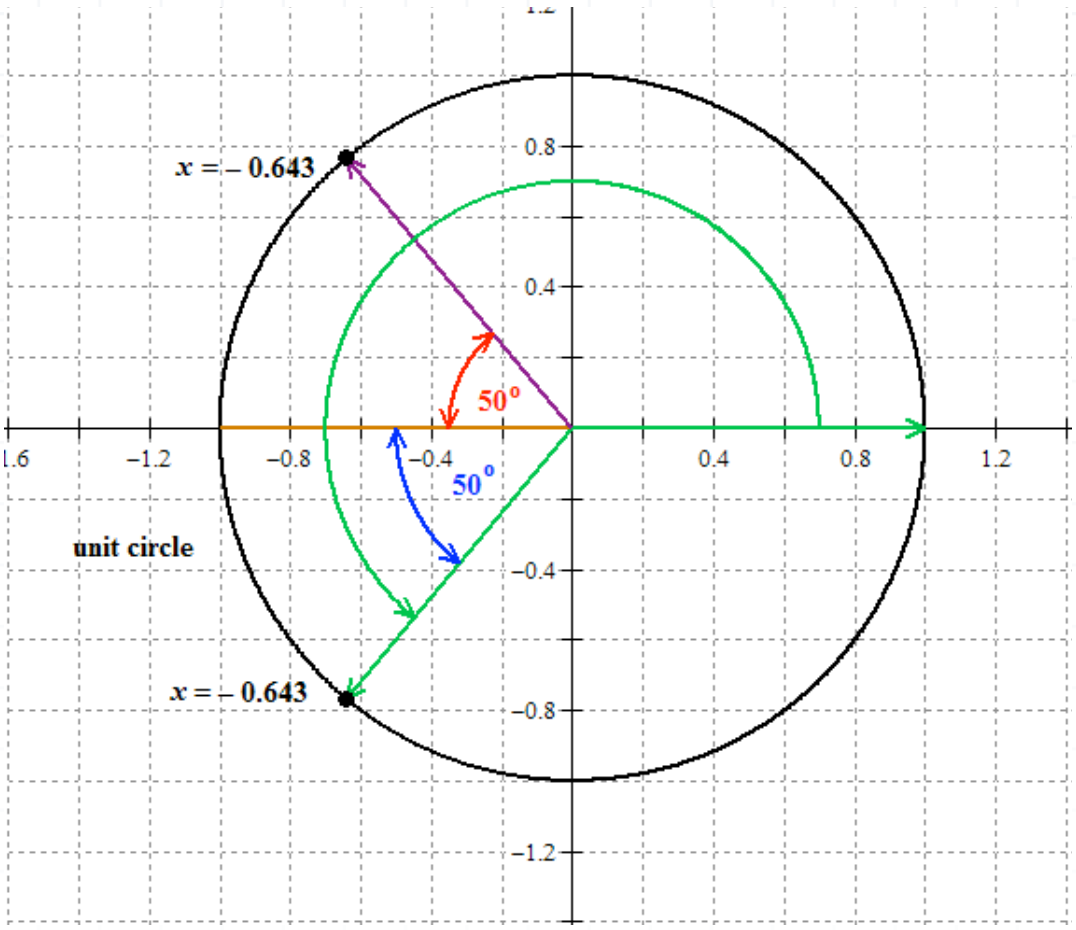

This other angle is $(180+50)^\circ = 230^\circ$. So the other set of angles that satisfy the equation is

$$\{230^\circ \pm (360^\circ)k:\ k = 0,1,2,\ldots\}$$

Combining these results, we find that the solution set of the equation $\cos\theta = -0.643$ is

$$\{130^\circ \pm (360^\circ)k:\ k = 0,1,2,\ldots\} \cup \{230^\circ \pm (360^\circ)k:\ k = 0,1,2,\ldots\}$$

Let's express these angles in radians. To do this, we'll use the fact that $180^\circ = \pi$:

$$130^\circ = 130^\circ\left(\frac{\pi}{180^\circ}\right)$$

$$130^\circ = \frac{130}{180}\pi$$

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$$130^\circ = \frac{13}{18}\pi$$

Similarly,

$$230^\circ = 230^\circ\left(\frac{\pi}{180^\circ}\right)$$

$$230^\circ = \frac{230}{180}\pi$$

$$230^\circ = \frac{23}{18}\pi$$

The solution set in radians is

$$\left\{\frac{13}{18}\pi \pm 2\pi k:\ k = 0,1,2,\ldots\right\} \cup \left\{\frac{23}{18}\pi \pm 2\pi k:\ k = 0,1,2,\ldots\right\}$$

Now suppose we want to find the solutions of an equation of the form

$$\sin\theta = b$$

for some constant b in the interval $[-1,1]$.

Example

Solve the equation $\sin\theta = 0.362$.

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If θ satisfies the equation $\sin\theta = 0.362$, then $\sin\theta$ is positive, which means the y-coordinate is positive, and therefore that the angle falls in the first or second quadrant.

If we use a calculator to find the inverse sine, we get

$$\sin\theta = 0.362$$

$$\arcsin(\sin\theta) = \arcsin(0.362)$$

$$\theta = \arcsin(0.362)$$

$$\theta \approx 21.2^\circ$$

This angle is 21.2° above the x-axis in the first quadrant, and by symmetry we can find the same y-value 21.2° off the x-axis in the second quadrant, which would be at

$$180^\circ - 21.2^\circ$$

$$158.8^\circ$$

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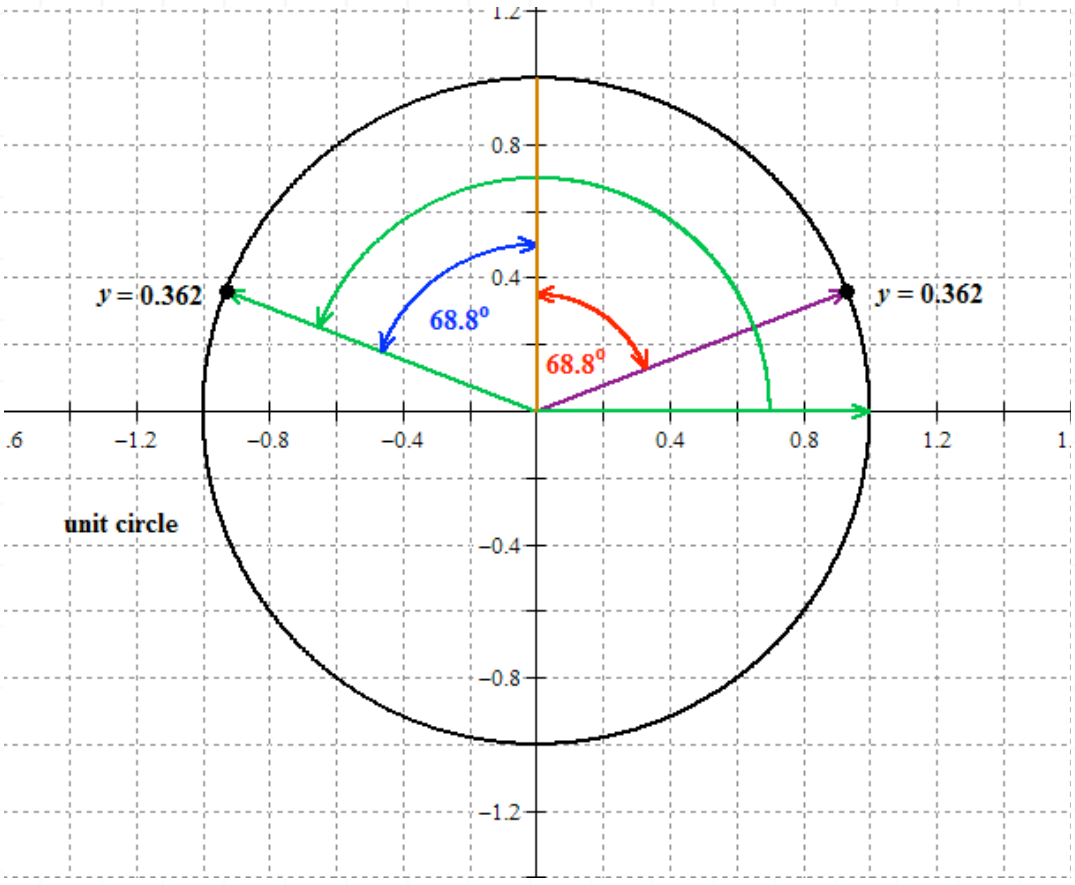

Once we include all angles that are coterminal with both of these, we can say that the values of θ that satisfy the equation are

$$\{21.2^\circ \pm (360^\circ)k: \ k = 0,1,2,\ldots\} \cup \{158.8^\circ \pm (360^\circ)k: \ k = 0,1,2,\ldots\}$$

Let's do another example.

Example

Solve the equation $\sin\theta = -0.519$.

Because the value of sine is negative, that means the value of y has to be negative, which means any angle that satisfies the equation will be in the third or fourth quadrant.

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If we use a calculator to find the inverse sine, we get

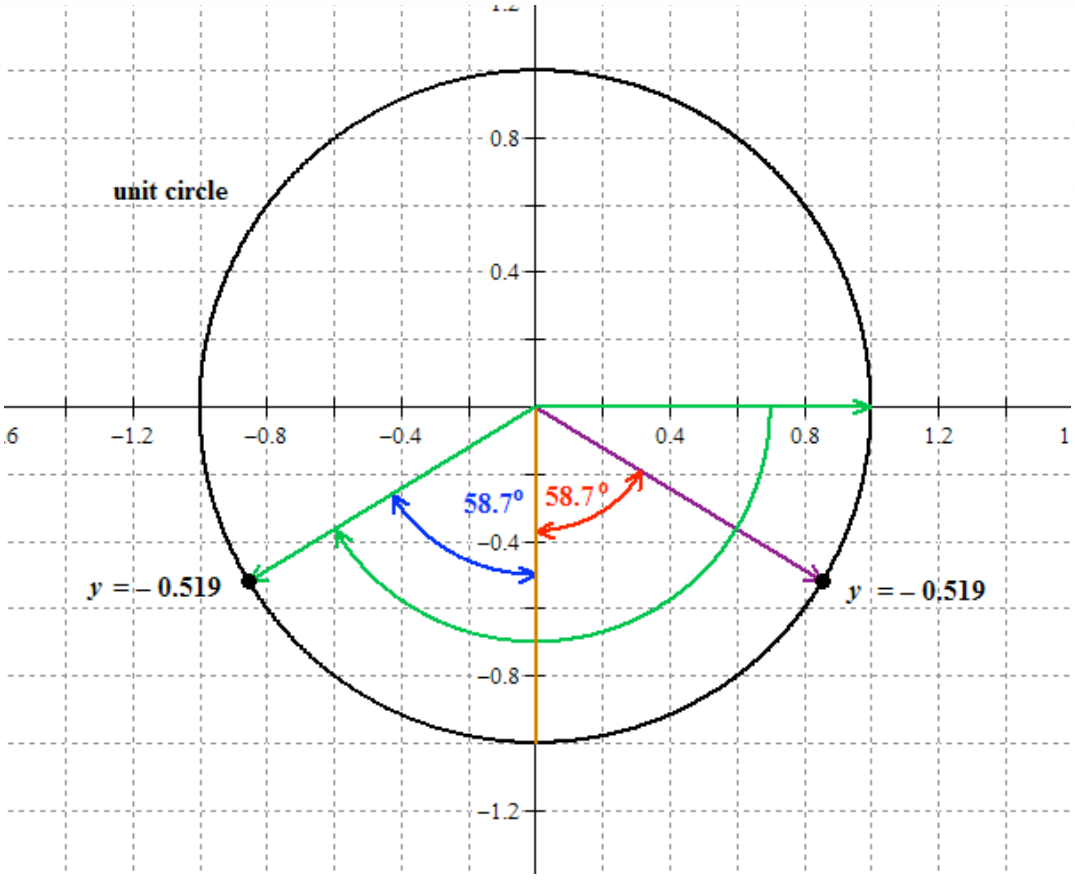
$$\sin\theta = -0.519$$

$$\arcsin(\sin\theta) = \arcsin(-0.519)$$

$$\theta = \arcsin(-0.519)$$

$$\theta \approx -31.3^\circ$$

Since this angle is 31.3° below the x-axis in the fourth quadrant, which means there's a symmetrical angle 31.3° degrees below the x-axis in the third quadrant. These two angles are 211.3° and 328.7°.



Therefore, the angles that satisfy the equation are all the angles coterminal with these two, which is the set

$$\{211.3^\circ \pm (360^\circ)k:\ k = 0,1,2,\ldots\} \cup \{328.7^\circ \pm (360^\circ)k:\ k = 0,1,2,\ldots\}$$

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